M13/5/MATHL/HP1/ENG/TZ2/XX/M

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

MARKSCHEME

May 2013

MATHEMATICS

Higher Level

Paper 1

17 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to Scoris instructions and the document "**Mathematics HL: Guidance for e-marking May 2013**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by Scoris.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets eg (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). A candidate should be penalized only once for a particular mis-read. Use the* ***MR*** *stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an* ***M*** *mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)****.*

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT****.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

SECTION A

1. $\left[\frac{1}{3}(x-2)^3+\ln x-\frac{1}{\pi}\cos \pi x\right]_{(1)}^{(2)}$ ***A1A1A1***

Note: Accept $\frac{1}{3}x^3-2x^2+4x$ in place of $\frac{1}{3}(x-2)^3$.

$=\left(0+\ln 2-\frac{1}{\pi}\cos 2\pi\right)-\left(-\frac{1}{3}+\ln 1-\frac{1}{\pi}\cos \pi\right)$ ***(M1)***

$=\frac{1}{3}+\ln 2-\frac{2}{\pi}$ ***A1A1***

Note: Award ***A1*** for any two terms correct, ***A1*** for the third correct.

Total [6 marks]

2. (a) $\det A=(5\times 2-3\times 3)=1$ ***(A1)***

$A^{-1}=\begin{pmatrix}2 & -3\\ -3 & 5\end{pmatrix}$ ***A1***

[2 marks]

(b) $\begin{pmatrix}2 & -3\\ -3 & 5\end{pmatrix}\begin{pmatrix}3 & -5\\ 2 & -3\end{pmatrix}$

$=\begin{pmatrix}2\times 3+(-3)\times 2 & 2\times(-5)+(-3)\times(-3)\\ (-3)\times 3+5\times 2 & (-3)\times(-5)+5\times(-3)\end{pmatrix}$ ***(M1)***

$=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ ***A2***

[3 marks]

Note: Award ***A2*** for all correct, ***A1*** for one error, ***A0*** otherwise.

Total [5 marks]

3. clear attempt at binomial expansion for exponent 5 ***M1***

$$2^5 + 5\times 2^4 \times (-3x) + \frac{5\times 4}{2}\times 2^3 \times (-3x)^2 + \frac{5\times 4\times 3}{6}\times 2^2 \times (-3x)^3$$

$$+\frac{5\times 4\times 3\times 2}{24}\times 2\times (-3x)^4 + (-3x)^5$$ ***(A1)***

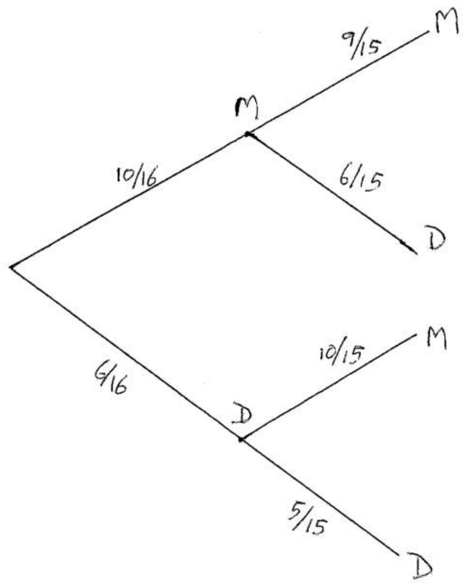
Note: Only award ***M1*** if binomial coefficients are seen.

$$= 32 - 240x + 720x^2 - 1080x^3 + 810x^4 - 243x^5$$ ***A2***

Note: Award ***A1*** for correct moduli of coefficients and powers. ***A1*** for correct signs.

Total [4 marks]

4. (a)



A1A1A1

[3 marks]

Note: Award ***A1*** for the initial level probabilities, ***A1*** for each of the second level branch probabilities.

(b) $\frac{10}{16}\times\frac{9}{15}+\frac{6}{16}\times\frac{5}{15}$ ***(M1)***

$=\frac{120}{240}\left(=\frac{1}{2}\right)$ ***A1***

[2 marks]

Total [5 marks]

5. (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x+\cos x)(\cos x - x\sin x) - x\cos x(1-\sin x)}{(x+\cos x)^2}$ ***M1A1A1***

Note:	Award ***M1*** for attempt at differentiation of a quotient and a product condoning sign errors in the quotient formula and the trig differentiations, ***A1*** for correct derivative of "*u*", ***A1*** for correct derivative of "*v*".

$= \frac{x\cos x + \cos^2 x - x^2\sin x - x\cos x\sin x - x\cos x + x\cos x\sin x}{(x+\cos x)^2}$ ***A1***

$= \frac{\cos^2 x - x^2\sin x}{(x+\cos x)^2}$ ***AG***

[4 marks]

(b) the derivative has value -1 ***(A1)***

the equation of the tangent line is $(y-0) = (-1)\left(x - \frac{\pi}{2}\right)\ \left(y = \frac{\pi}{2} - x\right)$ ***M1A1***

[3 marks]

Total [7 marks]

6. for the first series $\frac{a}{1-r} = 76$ ***A1***

for the second series $\frac{a}{1-r^3} = 36$ ***A1***

attempt to eliminate a *e.g.* $\frac{76(1-r)}{1-r^3} = 36$ ***M1***

simplify and obtain $9r^2 + 9r - 10 = 0$ ***(M1)A1***

Note:	Only award the ***M1*** if a quadratic is seen.

obtain $r = \frac{12}{18}$ and $-\frac{30}{18}$ ***(A1)***

$r = \frac{12}{18}\left(= \frac{2}{3} = 0.666\ldots\right)$ ***A1***

Note:	Award ***A0*** if the extra value of r is given in the final answer.

Total [7 marks]

7. (a) $|z_1| = \sqrt{10}$; $\arg(z_2) = -\dfrac{3\pi}{4}$ $\left(\text{accept } \dfrac{5\pi}{4}\right)$ ***A1A1***

[2 marks]

(b) $|z_1 + \alpha z_2| = \sqrt{(1-\alpha)^2 + (3-\alpha)^2}$ or the squared modulus ***(M1)(A1)***

attempt to minimise $2\alpha^2 - 8\alpha + 10$ or their quadratic or its half or its square root ***M1***

obtain $\alpha = 2$ at minimum ***(A1)***

state $\sqrt{2}$ as final answer ***A1***

[5 marks]

Total [7 marks]

8. (a) attempt at implicit differentiation ***M1***

EITHER

$$\frac{2x}{y} - \frac{x^2}{y^2}\frac{dy}{dx} - 2 = \frac{1}{y}\frac{dy}{dx}$$ ***A1A1***

Note: Award ***A1*** for each side.

$$\frac{dy}{dx} = \frac{\frac{2x}{y} - 2}{\frac{1}{y} + \frac{x^2}{y^2}} \left(= \frac{2xy - 2y^2}{x^2 + y} \right)$$ ***A1***

OR

after multiplication by y

$$2x - 2y - 2x\frac{dy}{dx} = \frac{dy}{dx}\ln y + y\frac{1}{y}\frac{dy}{dx}$$ ***A1A1***

Note: Award ***A1*** for each side.

$$\frac{dy}{dx} = \frac{2(x-y)}{1 + 2x + \ln y}$$ ***A1***

[4 marks]

(b) for $y = 1$, $x^2 - 2x = 0$

$x = (0 \text{ or}) \ 2$ ***A1***

for $x = 2$, $\dfrac{dy}{dx} = \dfrac{2}{5}$ ***A1***

[2 marks]

Total [6 marks]

9. (a) **EITHER**

$f(x)-1=\dfrac{1+3^{-x}}{3^x-3^{-x}}$ ***M1A1***

>0 as both numerator and denominator are positive ***R1***

OR

$3^x+1>3^x>3^x-3^{-x}$ ***M1A1***

Note:	Accept a convincing valid argument the numerator is greater than the denominator.

numerator and denominator are positive ***R1***
hence $f(x)>1$ ***AG***

[3 marks]

(b) one line equation to solve, for example, $4(3^x-3^{-x})=3^x+1$, or equivalent ***A1***
$(3y^2-y-4=0)$
attempt to solve a three-term equation ***M1***
obtain $y=\dfrac{4}{3}$ ***A1***

$x=\log_3\left(\dfrac{4}{3}\right)$ or equivalent ***A1***

Note:	Award ***A0*** if an extra solution for *x* is given.

[4 marks]

Total [7 marks]

10. (a) attempt at use of $\tan(A+B)=\dfrac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$ ***M1***

$\dfrac{1}{p}=\dfrac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\ \left(=\dfrac{1}{3}\right)$ ***A1***

$p=3$ ***A1***

Note:	the value of *p* needs to be stated for the final mark.

[3 marks]

(b) $\tan\left(\arctan\left(\dfrac{1}{2}\right)+\arctan\left(\dfrac{1}{5}\right)+\arctan\left(\dfrac{1}{8}\right)\right)=\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}=1$ ***M1A1***

$\arctan\left(\dfrac{1}{2}\right)+\arctan\left(\dfrac{1}{5}\right)+\arctan\left(\dfrac{1}{8}\right)=\dfrac{\pi}{4}$ ***A1***

[3 marks]

Total [6 marks]

SECTION B

11. (a) (i) $\vec{AB} = \vec{OB} - \vec{OA} = 5\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$ (or in column vector form) ***(A1)***

Note:	Award ***A1*** if any one of the vectors, or its negative, representing the sides of the triangle is seen.

$|\vec{AB}| = |5\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}| = \sqrt{30}$

$|\vec{BC}| = |-\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}| = \sqrt{11}$

$|\vec{CA}| = |-4\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}| = \sqrt{33}$ ***A2***

Note:	Award ***A1*** for two correct and ***A0*** for one correct.

(ii) **METHOD 1**

$$\cos \mathrm{BAC} = \frac{20+4+2}{\sqrt{30}\sqrt{33}}$$ ***M1A1***

Note:	Award ***M1*** for an attempt at the use of the scalar product for two vectors representing the sides AB and AC, or their negatives, ***A1*** for the correct computation using their vectors.

$$= \frac{26}{\sqrt{990}} \quad \left(= \frac{26}{3\sqrt{110}} \right)$$ ***A1***

Note:	Candidates who use the modulus need to justify it – the angle is not stated in the question to be acute.

METHOD 2

using the cosine rule

$$\cos \mathrm{BAC} = \frac{30+33-11}{2\sqrt{30}\sqrt{33}}$$ ***M1A1***

$$= \frac{26}{\sqrt{990}} \quad \left(= \frac{26}{3\sqrt{110}} \right)$$ ***A1***

[6 marks]

(b) (i) $\vec{BC} \times \vec{CA} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & -3 & 1 \\ -4 & 4 & 1 \end{vmatrix}$ ***A1***

$= \big((-3)\times 1 - 1\times 4\big)\boldsymbol{i} + \big(1\times(-4) - (-1)\times 1\big)\boldsymbol{j} + \big((-1)\times 4 - (-3)\times(-4)\big)\boldsymbol{k}$ ***M1A1***

$= -7\boldsymbol{i} - 3\boldsymbol{j} - 16\boldsymbol{k}$ ***AG***

(ii) the area of $\Delta\mathrm{ABC} = \frac{1}{2}\left|\vec{BC} \times \vec{CA}\right|$ ***(M1)***

$\frac{1}{2}\sqrt{(-7)^2 + (-3)^2 + (-16)^2}$ ***A1***

$= \frac{1}{2}\sqrt{314}$ ***AG***

[5 marks]

(c) attempt at the use of "$(\boldsymbol{r}-\boldsymbol{a})\bullet\boldsymbol{n}=0$" ***(M1)***

using $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$, $\boldsymbol{a}=\overrightarrow{\mathrm{OA}}$ and $\boldsymbol{n}=-7\boldsymbol{i}-3\boldsymbol{j}-16\boldsymbol{k}$ ***(A1)***

$7x+3y+16z=47$ ***A1***

Note:	Candidates who adopt a 2-parameter approach should be awarded, ***A1*** for correct 2-parameter equations for x, y and z; ***M1*** for a serious attempt at elimination of the parameters; ***A1*** for the final Cartesian equation.

[3 marks]

(d) $\boldsymbol{r}=\overrightarrow{\mathrm{OA}}+t\,\overrightarrow{\mathrm{AB}}$ (or equivalent) ***M1***

$\boldsymbol{r}=(-\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k})+t(5\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k})$ ***A1***

Note:	Award ***M1A0*** if "$\boldsymbol{r}=$" is missing.

Note:	Accept forms of the equation starting with B or with the direction reversed.

[2 marks]

(e) (i) $\overrightarrow{\mathrm{OD}}=(-\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k})+t(5\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k})$

statement that $\overrightarrow{\mathrm{OD}}\bullet\overrightarrow{\mathrm{BC}}=0$ ***(M1)***

$$\begin{pmatrix}-1+5t\\2-t\\3-2t\end{pmatrix}\bullet\begin{pmatrix}-1\\-3\\1\end{pmatrix}=0$$ ***A1***

$-2-4t=0$ or $t=-\frac{1}{2}$ ***A1***

coordinates of D are $\left(-\frac{7}{2},\frac{5}{2},4\right)$ ***A1***

Note:	Different forms of $\overrightarrow{\mathrm{OD}}$ give different values of t, but the same final answer.

(ii) $t<0 \Rightarrow$ D is not between A and B ***R1***

[5 marks]

Total [21 marks]

12. (a) by division or otherwise

$f(x) = 2 - \dfrac{5}{x+2}$ ***A1A1***

[2 marks]

(b) $f'(x) = \dfrac{5}{(x+2)^2}$ ***A1***

> 0 as $(x+2)^2 > 0$ (on D) ***R1AG***

Note:	Do not penalise candidates who use the original form of the function to compute its derivative.

[2 marks]

(c) $S = \left[-3, \dfrac{3}{2}\right]$ ***A2***

Note:	Award ***A1A0*** for the correct endpoints and an open interval.

[2 marks]

continued …

Question 12 continued

(d) (i) **EITHER**

rearrange $y = f(x)$ to make x the subject ***M1***

obtain one-line equation, *e.g.* $2x - 1 = xy + 2y$ ***A1***

$x = \dfrac{2y+1}{2-y}$ ***A1***

OR

interchange x and y ***M1***

obtain one-line equation, *e.g.* $2y - 1 = xy + 2x$ ***A1***

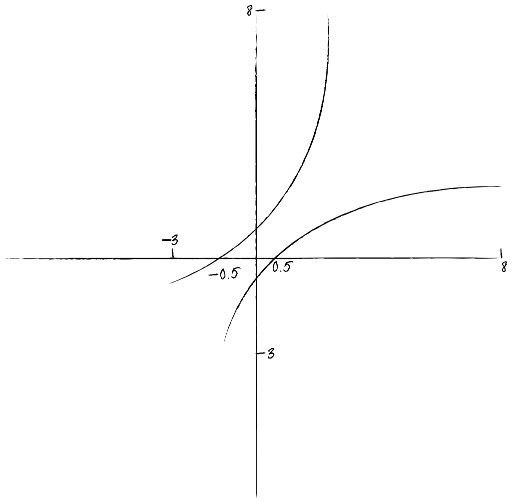
$y = \dfrac{2x+1}{2-x}$ ***A1***

THEN

$f^{-1}(x) = \dfrac{2x+1}{2-x}$ ***A1***

Note:	Accept $\dfrac{5}{2-x} - 2$

(ii), (iii)



A1A1A1A1

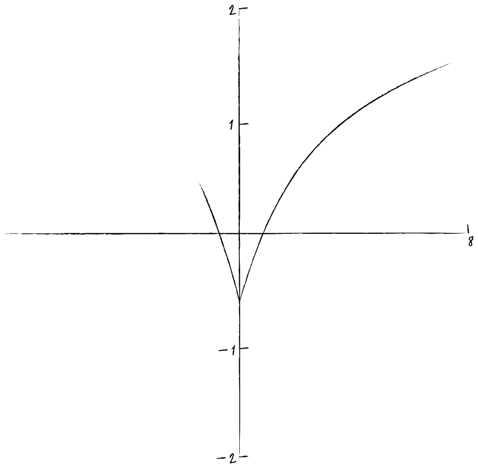
[8 marks]

Note:	Award ***A1*** for correct shape of $y = f(x)$. Award ***A1*** for x intercept $\dfrac{1}{2}$ seen. Award ***A1*** for y intercept $-\dfrac{1}{2}$ seen. Award ***A1*** for the graph of $y = f^{-1}(x)$ being the reflection of $y = f(x)$ in the line $y = x$. Candidates are not required to indicate the full domain, but $y = f(x)$ should not be shown approaching $x = -2$. Candidates, in answering (iii), can **FT** on their sketch in (ii).

continued …

Question 12 continued

(e) (i)



A1A1A1

Note:	***A1*** for correct sketch $x > 0$, ***A1*** for symmetry, ***A1*** for correct domain (from -1 to $+8$).

Note:	Candidates can **FT** on their sketch in (d)(ii).

(ii) attempt to solve $f(x) = -\frac{1}{4}$ ***(M1)***

obtain $x = \frac{2}{9}$ ***A1***

use of symmetry or valid algebraic approach ***(M1)***

obtain $x = -\frac{2}{9}$ ***A1***

[7 marks]

Total [21 marks]

13. (a) (i) $z_1 = 2\text{cis}\left(\frac{\pi}{6}\right)$, $z_2 = 2\text{cis}\left(\frac{5\pi}{6}\right)$, $z_3 = 2\text{cis}\left(-\frac{\pi}{2}\right)$ or $2\text{cis}\left(\frac{3\pi}{2}\right)$ ***A1A1A1***

Note:	Accept modulus and argument given separately, or the use of exponential (Euler) form.

Note:	Accept arguments given in rational degrees, except where exponential form is used.

(ii) the points lie on a circle of radius 2 centre the origin ***A1***

differences are all $\frac{2\pi}{3}$ (mod 2π) ***A1***

$\Rightarrow$ points equally spaced $\Rightarrow$ triangle is equilateral ***R1AG***

Note:	Accept an approach based on a clearly marked diagram.

(iii) $z_1^{3n} + z_2^{3n} = 2^{3n}\text{cis}\left(\frac{n\pi}{2}\right) + 2^{3n}\text{cis}\left(\frac{5n\pi}{2}\right)$ ***M1***

$= 2 \times 2^{3n}\text{cis}\left(\frac{n\pi}{2}\right)$ ***A1***

$2z_3^{3n} = 2 \times 2^{3n}\text{cis}\left(\frac{9n\pi}{2}\right) = 2 \times 2^{3n}\text{cis}\left(\frac{n\pi}{2}\right)$ ***A1AG***

[9 marks]

(b) (i) attempt to obtain **seven** solutions in modulus argument form ***M1***

$z = \text{cis}\left(\frac{2k\pi}{7}\right), k = 0, 1\ldots 6$ ***A1***

(ii) w has argument $\frac{2\pi}{7}$ and $1+w$ has argument ϕ,

then $\tan(\phi) = \dfrac{\sin\left(\frac{2\pi}{7}\right)}{1+\cos\left(\frac{2\pi}{7}\right)}$ ***M1***

$= \dfrac{2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{\pi}{7}\right)}{2\cos^2\left(\frac{\pi}{7}\right)}$ ***A1***

$= \tan\left(\frac{\pi}{7}\right) \Rightarrow \phi = \frac{\pi}{7}$ ***A1***

Note:	Accept alternative approaches.

continued ...

Question 13 continued

(iii) since roots occur in conjugate pairs, ***(R1)***

$z^7 - 1$ has a quadratic factor $\left(z - \text{cis}\left(\frac{2\pi}{7}\right)\right) \times \left(z - \text{cis}\left(-\frac{2\pi}{7}\right)\right)$ ***A1***

$= z^2 - 2z\cos\left(\frac{2\pi}{7}\right) + 1$ ***AG***

other quadratic factors are $z^2 - 2z\cos\left(\frac{4\pi}{7}\right) + 1$ ***A1***

and $z^2 - 2z\cos\left(\frac{6\pi}{7}\right) + 1$ ***A1***

[9 marks]

Total [18 marks]